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THE CANTOR–LEBESGUE PROPERTY

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Dedicated to the memory of Antoni Zygmund

ABSTRACT

If the terms of a trigonometric series tend to zero at each point of a set and if the smallest additive group containing that set has positive outer Lebesgue measure, then the coefficients of that series tend to zero. This result generalizes the well known Cantor–Lebesgue Theorem. Several other extensions of the Cantor–Lebesgue Theorem as well as some examples to demonstrate scope and sharpness are also given.

1. Introduction

The convergence of a trigonometric series with partial sums

$$S_n(x) = \sum_{k=-n}^{n} c_k e^{ikx}$$

implies that

$$\lim_{n\to\infty} \left(c_n e^{inx} + c_{-n} e^{-inx}\right) = 0. \tag{1}$$

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The classical Cantor–Lebesgue Theorem asserts that if $\{c_n\}_{n=-\infty}^{\infty}$ is a sequence of complex numbers with the property that (1) holds for every real number x in a set of positive measure then

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} c_{-n} = 0. \tag{2}$$

Thus, if the trigonometric series converges on a "large enough" set, the coefficients of the series necessarily tend to zero, so that the convergence in equation (1) cannot be due to cancellation between the terms $c_{-n}e^{-inx}$ and $c_n e^{inx}$. Theorem 1 extends this result by weakening the hypotheses in two ways. On the one hand, the condition that the set where relation (1) holds have positive measure may be considerably relaxed. On the other hand, the assumption (1) itself may be lightened to the assumption that certain blocks of the form

$$\sum_{\nu=n}^{n+r} \left(c_\nu e^{i\nu x} + c_{-\nu}e^{-i\nu x}\right)$$

tend to zero on the "large" set. The conclusion (2) will still follow. Indeed, it is natural to ask if (2) follows from the convergence of a trigonometric series on such a set E along a subsequence n_j. If $\sup(n_j - n_{j-1})$ is finite, then (2) follows as a special case of Theorem 1, which yields both extensions of the classical result to which we have alluded. We begin by extending our notion of "large" to include certain sets of measure zero. For any set E, denote by $\mathrm{gr}(E)$ the smallest additive subgroup of $\mathbf{R}$ (the real numbers) containing E. If (1) holds on some set E with $\mathrm{gr}(E)$ having positive outer measure, then (2) follows. We provide some examples below of sets E of measure zero satisfying this hypothesis. Below, $|E|$ will denote the Lebesgue measure of a set E.

THEOREM 1: *Let n_j be an increasing sequence of positive integers, and* $\limsup n_j - n_{j-1} =: r < \infty$. *Let*

$$A_j(x) := \sum_{\nu=n_j+1}^{n_{j+1}} c_\nu e^{i\nu x}, \quad A_{-j}(x) := \sum_{\nu=n_j+1}^{n_{j+1}} c_{-\nu} e^{-i\nu x}.$$

Suppose, for every x on a set E with $\mathrm{gr}(E)$ having positive outer measure, that

$$A_j(x) + A_{-j}(x) \to 0. \tag{3}$$

Then (2) obtains.

Remark: The hypothesis "gr(E) has outer measure zero" is not really much weaker than "gr(E) = $\mathbb{R}$." For suppose that (3) holds on some set E with gr(E) having positive outer measure. We will show that the set of all points on which (3) occurs(call it E') generates all of $\mathbb{R}$.

First of all, $E'(=\bigcap_{k=1}^{\infty}\bigcup_{n=1}^{\infty}\bigcap_{\nu=n}^{\infty}\{x\colon |A_\nu(x)+A_{-\nu}(x)|\le \frac{1}{k}\})$ is a Borel set, so that $G := \mathrm{gr}(E')$ is a Souslin (i.e., analytic) set, and consequently (see, e.g., page 482 of [Ku]), G is measurable. (Why use such a delicate argument here? Because it is possible for the group of a Lebesgue measurable set to be non-measurable.) But $E \subseteq E'$ implies $\mathrm{gr}(E) \subseteq G$ so G has positive measure. Since G is a group, $G = G - G$. From a theorem of Steinhaus (see, [Ox], p. 21 or [St]) showing that the set of differences of a set of positive measure contains an interval it follows that G contains some interval I. Next pick any two points of I and let d denote their difference. Then for any real $i \in I$ and any integer $n, nd + i \in G$, so that

$$\mathbb{R} = \bigcup_{n=-\infty}^{\infty} nd + I = \bigcup_{n=-\infty}^{\infty} \{nd + i\colon i \in I\} = G. \qquad \blacksquare$$

Again, the motivation for looking at $A_j + A_{-j}$ is that if $\{S_n\}$ is the sequence of symmetric partial sums of a complex trigonometrical series, and if $\{S_{n_j}\}$ is a subsequence of that sequence, then $S_{n_{j+1}} - S_{n_j} = A_j + A_{-j}$. In particular, convergence of the subsequence at a point x is a sufficient condition for $\lim_{j\to\infty} A_j(x) + A_{-j}(x) = 0$.

We will say that a set E has the **Cantor–Lebesgue Property** if (1) holding at each $x \in E$ implies (2). In other words, the classical Cantor–Lebesgue Theorem asserts that sets of positive Lebesgue measure have the Cantor–Lebesgue Property and specializing Theorem 1 to the case of block length $r = 1$ amounts to saying that any set generating an additive group with positive outer measure has the Cantor–Lebesgue Property.

An easy computation shows that a set E has the Cantor–Lebesgue property if and only if every pair of real sequences $\{a_n\}$, $\{b_n\}$ satisfying

$$(1') \qquad \lim_{n\to\infty} \big(a_n \cos(nx) + b_n \sin(nx)\big) = 0 \quad \text{for all } x \in E$$

necessarily satisfies

$$(2') \qquad \lim_{n\to\infty} \rho_n = 0,$$

where $\rho_n := \sqrt{a_n^2 + b_n^2}$. We thus can restate Theorem 1 as

COROLLARY 1′: *Let*

$$C_j(x) := \sum_{\nu=n_j+1}^{n_{j+1}} (a_\nu \sin(\nu x) + b_\nu \cos(\nu x)).$$

Suppose that $\limsup n_j - n_{j-1} < \infty$ *and that* $\lim_{j\to\infty} C_j(x) = 0$ *for all* x *in a set* E *with* $\mathrm{gr}(E)$ *having positive outer measure. Then* (2′) *obtains.*

Corollary 2 below gives a version of Theorem 1 adapted to the situation involving Cesaro summability of order α, where $\alpha > -1$. Examples 3 and 4 give some insight into the extent to which Theorem 1 generalizes the classical result. In particular, sets of second Baire category and many sets of Cantor type have group equal to the real numbers. Example 5 shows that the very strong growth restriction on the block length in Theorem 1 cannot be relaxed at all. Example 6 shows that the Cantor–Lebesgue Property is not enjoyed by all uncountable sets. Proposition 7 shows that Theorem 1 is not a characterization; a set may have the Cantor–Lebesgue Property even though its group has zero measure. Theorem 8 helps unify the Cantor–Lebesgue theory and the theory of absolute convergence of Fourier series by simultaneously generalizing a result of Zygmund(which generalizes a theorem of Niemytski which in turn generalizes the Denjoy–Lusin Theorem) on absolute convergence and the Cantor–Lebesgue Theorem. Finally we define the Absolute Convergence Property and comment on its relation to the Cantor–Lebesgue Property.

We were initially attracted to this subject by the very interesting Remark 2 on page 561 of a nice survey paper by R. Cooke [Co].

2. Results

Proof of Theorem 1: First, we can without loss of generality assume that $\mathrm{gr}(E) = \mathbb{R}$, by replacing E by the set E' mentioned in the Remark above if necessary.

Next, we reduce the proof to the case of bounded coefficients c_ν. If (3) did not imply (2) for a particular sequence $\{c_\nu\}$, then we would construct a new sequence $\{d_\nu\}$ bounded by 1 for which (3) would hold but (2) would not, as follows. For each ν satisfying $n_j + 1 \le |\nu| \le n_{j+1}$ $(j = 1, 2, \dots)$, let

$$d_\nu = \frac{c_\nu}{\max\{1, \max_{n_j+1\le|k|\le n_{j+1}}\{|c_k|\}\}}.$$

Then (3) still holds with $\{c_\nu\}$ replaced by $\{d_\nu\}$ although

$$0 < \limsup_{|\nu|\to\infty} |d_\nu| = \min\{\limsup_{|\nu|\to\infty} |c_\nu|, 1\} \le 1.$$

Also note that (2) must also fail for $\{d_\nu\}$ if it does for $\{c_\nu\}$. Since the $\{d_\nu\}$ are bounded, there is an $s \in \{0, 1, \ldots, r-1\}$ such that there is a subsequence on which $A_j(x)$ and $A_{-j}(x)$ have exactly $s+1$ terms. Along this subsequence

$$A_j(x) = P_{n_j}(x)e^{i(n_j+1)x} \qquad A_{-j}(x) = Q_{n_j}(x)e^{-i(s+n_j+1)x}$$

where P_{n_j} and Q_{n_j} are trigonometric polynomials of degree s.

Think of the coefficients of the polynomial pair (P_{n_j}, Q_{n_j}) as a point of $\mathbb{C}^{2s+2}$. Since this sequence of vectors lies in a compact set, it has a convergent subsequence. In other words, there is a subsequence of $\{n_j\}$, call it $\{n_j\}$ again, and trigonometric polynomials $P(x)$ and $Q(x)$ so that for all x

$$\lim_{j\to\infty} P_{n_j}(x) = P(x) \quad \text{and} \quad \lim_{j\to\infty} Q_{n_j}(x) = Q(x).$$

Along this sub-subsequence we have from (3) that

$$\lim_{j\to\infty} P(x)e^{i(n_j+1)x} + Q(x)e^{-i(n_j+s+1)x} = 0$$

on E. Further, if (2) does not hold, we can assume for this choice of s that the limiting polynomials are not identically zero. Thus (assuming without loss that E does not contain any zeros of $P(x)$ or $Q(x)$) we have that on E

$$\lim_{j\to\infty} \frac{P(x)}{Q(x)} e^{i(2n_j+s+2)x} = -1.$$

In particular, the ratio of 2 successive terms of the left side must tend to 1. In other words

$$\lim_{j\to\infty} e^{2i(n_{j+1}-n_j)x} = 1. \tag{4}$$

Since any element of $\mathrm{gr}(E)$ may be obtained from finite sums and differences of elements in E, relation (4) holds in fact for all elements of $\mathrm{gr}(E)$, by the multiplicative property of the exponential function. But $\mathrm{gr}(E) = \mathbb{R}$ so that, in particular, (4) holds on the interval $[0, 2\pi]$. Hence by Lebesgue's Dominated Convergence Theorem

$$\lim_{j\to\infty} \int_0^{2\pi} e^{2i(n_{j+1}-n_j)x} dx = 2\pi,$$

which is impossible since the integral inside the limit is 0. ∎

COROLLARY 2: *Suppose* $c_ne^{inx}+c_{-n}e^{-inx}$ *is* (C,α) *summable to 0 on* $E, \alpha > -1$, *and that* $\mathrm{gr}(E)$ *has positive outer measure. Then*

$$|c_n| + |c_{-n}| = o(n^\alpha).$$

Proof: It suffices to consider the real part. If $T_n = \{a_n \cos(nx) + b_n \sin(nx)\}$ is (C,α) summable to 0, then ([Zy], p. 78) $T_n = o(n^\alpha)$. Replacing a_n, b_n, ρ_n with (respectively) a_n/n^α, b_n/n^α, ρ_n/n^α , we see that it suffices to prove that $T_n \to 0$ on E and $\mathrm{gr}(E)$ having positive outer measure imply that $\rho_n \to 0$. ∎

Remark: One might suppose that Corollary 2 could be strengthened so as to conclude that ρ_n is (C,α) summable to zero. Such a conclusion is in general false, however. Consider $T_0(x) = \frac{1}{2}$; $T_n(x) = \cos(nx)$ $(n > 0)$. Then $\{T_n(x)\}$ is $(C,1)$ summable to 0 for all $x \neq 0$ on $[-\pi,\pi)$ but $\rho_n = \sqrt{1/2}$ $(n > 0)$ so that $\{\rho_n\}$ is $(C,1)$ summable to $\sqrt{1/2}$ (and not 0). ∎

That Theorem 1 actually generalizes the classical result may be seen by considering the following examples of sets of measure 0 which generate a group of positive measure, namely, the real numbers.

Example 3: The set of differences of any set of the second category (in the sense of Baire) contains an interval [Ox], p.21. There are of course sets of the second category that have measure zero. Young [Yo] had obtained the generalization of the Cantor–Lebesgue Theorem when E is the complement of a set of the first category.

Example 4: A set E of Cantor type with constant ratio of dissection ζ $(0 < \zeta < 1)$ is defined as

$$E := \left\{ 2\pi(1-\zeta)\sum_{k=1}^{\infty} \epsilon_k \zeta^{k-1} \colon \epsilon_k \in \{0,1\} \right\}.$$

The classical Cantor set corresponds to $\zeta = 1/3$. All such sets have measure 0. Nevertheless, for any such set E, $\mathrm{gr}(E) = \mathbb{R}$ [Ka], p. 566.

Remarks: One might wonder if the hypothesis that $\mathrm{gr}(E)$ have positive outer measure gives Theorem 1 any more scope than the stronger, but simpler, hypothesis that the set of differences of elements of E contains an interval. However, when $\zeta < \frac{1}{3}$, the set of differences does not fill an interval.

We remark in passing that "thin" Cantor sets, obtained by allowing the ratio of dissection to decrease to 0 (see [Zy], p.195, for the definition of sets of Cantor type with variable ratio of dissection), will generate groups of measure zero [KS], p. 103. See [Ka] for generalizations. We do not know if thin Cantor sets have the Cantor–Lebesgue Property.

There is yet another kind of "thin" set whose group is $\mathbb{R}$. These sets were discovered independently by Mirimanoff who called them perfect sets of the "1[re] espèce," and by Denjoy who called them sets "présentant le caractère (A)." See [Mi] for a definition. ∎

Suppose now that $\limsup(n_k - n_{k-1}) = \infty$. The following example (cf. [Ba], IV, §26, exercise 9) shows how badly the conclusion of Theorem 1 can fail.

Example 5: Let n_j be an increasing sequence of integers such that $\limsup(n_j - n_{j-1}) = \infty$ and $\epsilon > 0$ be given. Then we construct a trigonometric series $S(x)$ such that $S_{n_j}(x)$ converges uniformly on $[0, 2\pi - \epsilon]$ and $\limsup(|c_j| + |c_{-j}|) = \infty$.

Let $0 \le f_k$ be a C^∞ function with support in $(2\pi - \epsilon, 2\pi)$ and $\frac{1}{2\pi}\int f_k = k$ and let $P_k(x) = \sum_{\nu=-N_k}^{N_k} c_\nu^{(k)} e^{i\nu x}$ be a partial sum of the Fourier expansion of f_k chosen so that for every $x \in [0, 2\pi - \epsilon]$ we have $|P_k(x)| < 1/k^2$. Find a gap big enough to hold P_1, i.e., find the first n_j satisfying $n_j - n_{j-1} > 2N_1$; and then multiply P_1 by an appropriate character to slide it into the gap, i.e., $e^{i(n_{j-1}+1+N_1)x} P_1(x)$ has all of its frequencies in the interval $[n_{j-1}+1, n_j]$. Next find a gap big enough to hold P_2 and slide P_2 into that gap, and so on. The resultant series will have the form $\sum_{k=1}^{\infty} e^{im_k x} P_k(x)$. Our construction guarantees that any subsequential partial sum will consist only of complete unbroken blocks. By the Weierstrass M-test, $S_{n_k}(x)$ converges uniformly on $[0, 2\pi - \epsilon]$ but (since $c_0^{(k)} = k$) the coefficients of S are unbounded.

One might be tempted to conclude that if (1) holds on an uncountable set, then (2) must follow. However:

Example 6: There is an uncountable set not enjoying the Cantor–Lebesgue Property. For each $i = 1, 2, \ldots$, let $n_i := \frac{i(i+1)}{2}$, so that $n_i - n_{i-1} = i$. Then the sequence $\{\sin(2^{n_i} x)\}$ converges to zero uniformly on the uncountable set $E := \{2\pi(0.\epsilon_1 0 \epsilon_2 00 \epsilon_3 000 \epsilon_4 \ldots) \colon \epsilon_i \in \{0,1\} \text{ for all } i\}$, where $0.\epsilon_1 0 \epsilon_2 00 \cdots :=$

$\sum_{i=1}^{\infty} \epsilon_i 2^{n_i}$ is a binary decimal expansion. For notice that

$$2^{n_i}(0.\epsilon_1 0 \epsilon_2 00\dots) \equiv 0.\underbrace{0\dots0}_{i \text{ zeros}}\epsilon_{i+1}\dots \quad (\text{mod } 1);$$

so whenever $x = 2\pi(0.\epsilon_1 0 \epsilon_2 \dots) \in E$,

$$|\sin(2^{n_i}x)| = |\sin(2\pi(0.\epsilon_{i+1}0\dots)2^{-i})| \leq |2\pi(0.\epsilon_{i+1}0\dots)2^{-i}| \leq 2\pi 2^{-i}.$$

Remark: The gaps in the expansion defining E were picked with simplicity in mind. What really matters is that the gap length $n_i - n_{i-1}$ is nondecreasing to ∞. In particular a "fatter" example can be created by having the gap lengths tend to ∞ very slowly. Nevertheless, any such example will still generate a group of measure 0 by virtue of Theorem 1. ∎

Example 7: A set may enjoy the Cantor–Lebesgue Property even though it generates a group of measure zero. Any classical Salem set will do. On page 132 of [Va] the author observes that such a set $K \subset [0, 2\pi)$ has the following properties: It is closed, it generates a group of measure 0, there is a positive measure $d\mu$ supported on K with $\int_0^{2\pi} d\mu = \int_K d\mu = 1$ and $\hat{\mu}(n) \to 0$ as $|n| \to \infty$. As in the proof of Theorem 1 we may assume that $\{c_n\}$ is a bounded sequence and that (1) holds for each $x \in K$. The formula $\hat{\mu}(-m) = \frac{1}{2\pi}\int e^{imx} d\mu(x)$ and the identity $|z|^2 + |w|^2 = |z+w|^2 - 2\Re(z\bar{w})$ lead to

$$\begin{aligned} |c_n|^2 + |c_{-n}|^2 &= \int_K (|c_n|^2 + |c_{-n}|^2) d\mu \\ &= \int_K |c_n e^{inx} + c_{-n}e^{-inx}|^2 d\mu - 4\pi\Re(c_n \bar{c}_{-n}\hat{\mu}(-2n)). \end{aligned}$$

As $n \to \infty$, the first term on the right tends to 0 by Lebesgue's Bounded Convergence Theorem while the other tends to 0 since $\{c_n\}$ is bounded and $\hat{\mu}(n) \to 0$ as $|n| \to \infty$. Thus the left side tends to 0, which is equivalent to (2).

Remark: On page 370 of [Zy] appears the following interesting theorem. Given any set E of positive measure and any integer $m \geq 1$, there is a positive number $\delta = \delta(E, m)$ such that for every sum $c_1 e^{ip_1 x} + c_2 e^{ip_2 x} + \dots + c_m e^{ip_m x}$ with integral $p_1 < p_2 < \dots < p_m$ we have

$$\int_E |\sum_{s=1}^{m} c_s e^{ip_s x}|^2 dx \geq \delta \sum_{s=1}^{m} |c_s|^2.$$

Roger Cooke observed that this result yields a block version of the Cantor–Lebesgue Theorem similar to Theorem 1, but only for sets of positive measure. Zygmund's result can be generalized to prove a block version of the Cantor–Lebesgue Theorem for Salem sets. ∎

The next Theorem extends a result of Zygmund given in [Zy] (cf. VI, 1.12) on the absolute convergence of trigonometric series. Recall that the problem of interest in this connection is to determine the convergence behavior of

$$\sum_{k=1}^{\infty}(|a_k| + |b_k|)$$

when

$$\sum_{k=1}^{\infty} |a_k \cos(kx) + b_k \sin(kx)| < \infty \quad \text{for all } x \in E.$$

The result alluded to above asserts that when $\mathrm{gr}(E) = \mathbb{R}$, convergence of the latter series implies that of the former. (For the genealogy of Zygmund's Theorem, see page 380 of [Zy].) Theorem 8 may be construed as a generalization of this fact. It also yields our extension of the Cantor–Lebesgue Theorem (Theorem 1) in the case $r = 1$ as is remarked after Lemma 9 below.

THEOREM 8: *Let $a_k \geq 0$ and $\sum a_k = \infty$. Suppose that there exist sequences $\{n_k\}$, $\{\theta_k\}$ such that for all x in a set E,*

$$\sum_{k=1}^{N} a_k |\sin(n_k x + \theta_k)| = o(\sum_{k=1}^{N} a_k).$$

Then $\mathrm{gr}(E)$ has measure 0.

We begin with a lemma that shows Theorem 8 to be an extension of the Cantor–Lebesgue Theorem.

LEMMA 9: *If E does not have the Cantor–Lebesgue property, then there exist sequences $\{n_k\}$, $\{\theta_k\}$ such that $\sin(n_k x + \theta_k) \to 0$ on E.*

Proof: We suppose that (1′) holds for all $x \in E$ and that $\limsup \rho_n = 2c > 0$. There is then a subsequence with $\rho_{n_k} > c$ for all k. For each k, pick θ_k satisfying $\cos(\theta_k) = b_{n_k}/\rho_{n_k}$, and $\sin(\theta_k) = a_{n_k}/\rho_{n_k}$. Then

$$\begin{aligned} |\sin(n_k x + \theta_k)| &= \left| \frac{1}{\rho_{n_k}} [a_{n_k} \cos(n_k x) + b_{n_k} \sin(n_k x)] \right| \\ &\leq \frac{1}{c} \left| a_{n_k} \cos(n_k x) + b_{n_k} \sin(n_k x) \right|, \end{aligned}$$

the last expression tending to zero for all x in E by hypothesis. ∎

Remarks: This result can be strengthened as follows. First, we can assume that the θ_k converge to θ (by passing to a subsequence). Then write $\sin(n_k x+\theta) = \sin(n_k x+\theta_k)+\{\sin(n_k x+\theta)-\sin(n_k x+\theta_k)\}$, and note that sine is a uniformly continuous function to get that $\sin(n_k x+\theta)$ converges to 0 on E. Now choose a subsequence of the n_k so that $m_k := n_k - n_{k-1}$ increases to infinity. Writing

$$\sin(m_k x) = \sin((n_k x+\theta)-(n_{k-1}x+\theta))$$

and applying the sine addition formula then shows that we can conclude that for E not having the Cantor–Lebesgue property, there is a sequence m_k so that $\sin(m_k x)\to 0$ for every x in E. If E is also closed, it will be called an **A-set**, as in [BKL].

Note that by virtue of Lemma 9, Theorem 1 in the case $r=1$ follows as a Corollary of Theorem 8 by setting all $a_k = 1$. ∎

Proof of Theorem 8: Define

$$E[J] := \left\{x\colon\ \sum_{k=1}^{N} a_k|\sin(n_k x+\theta_k)| < \frac{2}{\pi(2r+1)}\sum_{k=1}^{N} a_k,\ \text{for all } N\ge J\right\},$$

where r is a positive integer. Let $E^{(r)}$ be the set of numbers of the form $s_1+\cdots+s_r-t_1-\cdots-t_r$, where $s_i,\ t_i$ are in E. Since $E=\bigcup E[J]$, and for all J, $E[J]\subset E[J+1]$, $E^{(r)}=\bigcup E[J]^{(r)}$. Let $E_n=\{s_1\pm\ldots\pm s_m\colon\ m\le n,\ s_i\in E\}$ and observe that $\mathrm{gr}(E)=\bigcup E_n$. Also, note that

$$E_n=\bigcup_{m=1}^{n}\bigcup_{r=0}^{m} A_{rm} := \bigcup_{m=1}^{n}\bigcup_{r=0}^{m}\{s_1+\cdots+s_r-s_{r+1}-\cdots-s_m\colon s_i\in E\},$$

and $E^{(m)}=A_{rm}-A_{rm}$ for any r (where $S-S$ is the set of all differences of elements of the set S). We may thus conclude that for measurable E, $|\mathrm{gr}(E)|>0$ if and only if $|E_n|>0$ for some n if and only if $|E^{(n)}|>0$ for some n. (Curiously, we neither assert nor require that $\bigcup E^{(n)}=\mathrm{gr}(E)$.) It thus suffices to show that $m=|E[J]^{(r)}|=0$, for every r.

Suppose that $m>0$. Let $w\in E[J]^{(r)}$, so that $w=s_1+\cdots+s_r-t_1-\cdots t_r$, with $s_i,\ t_i\in E[J]$. Since the sine addition formula implies $|\sin(\psi_1+\cdots+\psi_n)|\le\sum|\sin(\psi_j)|$, we have

$$|\sin(n_k w)|\le\sum_{j=1}^{r}|\sin(n_k s_j+\theta_k)|+|\sin(n_k t_j+\theta_k)|. \tag{5}$$

With $N \geq J$, we multiply equation (5) through by a_k, sum from $k = 1$ to N and can obtain

$$\sum_{k=1}^{N} a_k |\sin(n_k w)| \leq 2r \cdot \frac{2}{(2r+1)\pi} \sum_{k=1}^{N} a_k \tag{6}$$

by changing the order of summation and applying the defining relation of $E[J]$ $2r$ times. Setting

$$I_k = \int_{E[J]^{(r)}} |\sin(n_k x)|,$$

we obtain from equation (6)

$$\sum_{k=1}^{N} a_k I_k \leq 2mr \cdot \frac{2}{(2r+1)\pi} \sum_{k=1}^{N} a_k. \tag{7}$$

Since for any measurable set F, $\lim_{n\to\infty} \int_F |\sin(nx)| dx = \frac{2}{\pi}|F|$ (see, e.g., Theorem 4.15 on page 49 of [Zy]), there is k_0 so that

$$I_k \geq \frac{2}{\pi} m(1-\epsilon), \quad \text{for } k \geq k_0, \tag{8}$$

where we take $\epsilon = 1/(2(2r+1))$. Now choose $N \geq k_0$ so large that

$$\sum_{k<k_0} a_k < \epsilon \sum_{k=1}^{N} a_k. \tag{9}$$

We then compute

$$\begin{aligned}
&\frac{2}{\pi} m \sum_{k=1}^{N} a_k = \frac{2}{\pi} m \sum_{k<k_0} a_k + \frac{2}{\pi} m \sum_{k\geq k_0} a_k \leq \quad \text{(by (8))} \\
&\frac{2}{\pi} m \sum_{k<k_0} a_k + \sum_{k\geq k_0} (a_k(I_k + \frac{2}{\pi} m\epsilon)) < \quad \text{(by the positivity of } m) \\
&\frac{2}{\pi} m \sum_{k<k_0} a_k + \sum_{k\geq k_0} (a_k(I_k + \frac{2}{\pi} m\epsilon)) + \sum_{k<k_0} a_k I_k \leq \quad \text{(by (9))} \\
&\frac{2}{\pi} m\epsilon \sum_{k=1}^{N} a_k + \frac{2}{\pi} m\epsilon \sum_{k=1}^{N} a_k + \sum_{k=1}^{N} a_k I_k \leq \quad \text{(by (7))} \\
&\frac{2}{\pi} m(\frac{2r}{2r+1} + 2\epsilon) \sum_{k=1}^{N} a_k = \frac{2}{\pi} m \sum_{k=1}^{N} a_k,
\end{aligned}$$

the last equality by the choice of ϵ. This contradiction shows that $m = 0$, as desired, for any choice of J, r. This completes the proof. ∎

Say that a set has the **Absolute Convergence Property** if the absolute convergence of a trigonometric series at each point of that set implies that the sum of the absolute values of the coefficients is also convergent. On the other hand, a closed set E is an N-set if there is a sequence $\{c_n\}$ so that $\sum |c_n \sin nx|$ converges for all $x \in E$, although $\sum |c_n| = \infty$. Notice that N-sets do not have the Absolute Convergence Property.

Remarks: Theorem 8 might lead one to conjecture that a set has the Absolute Convergence Property if and only if it has the Cantor–Lebesgue Property. This, however, is not the case. If E is an A-set (cf. Remarks following Lemma 9), there is a sequence $\{n_k\}$ such that for every $x \in E$, $|\sin n_k x| \to 0$. By Lebesgue's Bounded Convergence Theorem, for every measure μ in the dual of $C(E)$, $\lim_{k\to\infty}\langle|\sin n_k x|, \mu\rangle = \lim_{k\to\infty}\int|\sin n_k x|d\mu = 0$, i.e., 0 is a weak limit of $\{|\sin n_k x|\}_{k=1}^{\infty}$ in $L^{\infty}(E)$. Mazur's theorem then asserts that 0 is in the norm closure of the convex hull of this sequence [KL], p.169. We thus can find nonnegative numbers a_{k1} so that $\sum_{k=1}^{\infty} a_{k1} = 1$ and $\sup_E \sum_{k=1}^{\infty} a_{k1}|\sin n_k x| < 2^{-1}$. Moreover, for each $j > 1$, 0 is also a weak limit of the sequence $\{|\sin n_k x|\}_{k=j}^{\infty}$, so that we may find a nonnegative sequence a_{kj} so that $\sum_{k=j}^{\infty} a_{kj} = 1$ and $\sup_E \sum_{k=j}^{\infty} a_{kj}|\sin n_k x| < 2^{-j}$. Thus, for every x in E, $\sum_{k=1}^{\infty} c_k|\sin n_k x| < 1$ where $c_k := \sum_{j=1}^{k} a_{kj}$. Since $\sum c_k = \sum_j \sum_k a_{kj}$ is clearly divergent, we conclude that E is an N-set. The family of A-sets is complete Σ_2^1, while the family of N-sets is not [BKL]. Thus there is a closed set having the Cantor–Lebesgue Property, but not the Absolute Convergence Property.

The class of Dirichlet sets ([KL], p.338) is related to these sets, and is rather simple topologically. ∎

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